Practical Stability of High-Eccentricity Orbits Quasi-Normal to the Ecliptic

MARC L. RENARD*
Carnegie-Mellon University, Pittsburgh, Pa.

A critical problem in the analysis of orbits of high eccentricity ($0.9 \leqslant e \leqslant 0.95$) is the determination of their practical stability, i.e., the evolution of the height of perigee over some prescribed lifetime. If the altitude of perigee remains larger than a specified value (close to the initial one) over the time considered, the corresponding launch time is labeled a stable point on the launch window map. This paper examines the practical stability of highly eccentric orbits quasi-normal to the ecliptic. The significant parameters of the problem and their influence on the orbital lifetime are discussed with reference to approximate stability criteria. The effects of inclination of the moon's orbit on the ecliptic and of a slight departure of the satellite orbit from normal to the ecliptic are determined. Good agreement is found between these predictions and the results of a detailed stability study for an actual satellite of eccentricity $e \cong 0.95$.

Nomenclature

a,e = semimajor axis and eccentricity of orbit, respectively
h,i = altitude and inclination, respectively
p = parameter
r = radius vector
t = time
w = factor
A = amplitude
I_M = inclination of moon's orbit on the ecliptic
RA_α,RA_ϵ = right ascension and celestial longitude, respectively
T^* = predicted lifetime
X,Y,Z = inertial axes
δ = change per satellite orbital period
ϵ = $1 - e^2$
ϵ_* = inclination of ecliptic on equator
τ = orbital period
ξ_1,ξ_2 = projection of unit to disturbing body on line of apsides (1) and semilatus rectum (2)
ω = argument of perigee
$\bar{\omega}_\sim$ = plane
$\omega_\sim$ = argument of perigee referred to $\bar{\omega}_\sim$
ψ = angle
γ = flight-path angle at injection
γ_s = angle between spin axis and injection velocity, in orbital plane
Ω = longitude of nodes
$\langle\ \rangle$ = averaged
$\mathbf{1}$ = unit vector

Subscripts

d = referred to orbital plane
A = apogee
E,M,S = Earth, moon, and sun, respectively
INT,LR = intermediate-range and long-range, respectively
P,p = perigee
SR,VLR = short-range and very-long-range, respectively
$*$ = projected on ecliptic
o = initial
α = equatorial
ϵ = ecliptic

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* Chairman, Applied Space Sciences Program, and Associate Professor, Electrical Engineering Department. Member AIAA.

Introduction and Purpose of the Study

ORBITS of high eccentricity ($0.9 \leqslant e \leqslant 0.95$) about the Earth present very distinctive features due to the gravitational effects of sun and moon. A critical problem is what may be called their practical stability, which is concerned with the evolution of the height of perigee h_P over some prescribed lifetime L (characteristically of the order of one year). If h_P remains above some critical value $h_P{}^*$, equal to or very slightly lower than the low value at injection, the orbit is labeled stable as well as the corresponding point of the plane (day of launch, hour of launch). The set of stable points defines the launch window map.

The systematic investigation of the stability of highly eccentric orbits was first carried out at NASA,[1–7] in the period 1963–1965, for satellites of the Interplanetary Monitoring Probes (IMP) series. Studies were also made in Europe.[8–13] The methods used were: a) integration on a digital computer of the trajectory over the lifetime by, Encke's method[4–7] inside a limiting contour determined by Halphen-Goryachev's method; or a method of variation of parameters,[8–13] and b) analog integration of M. Moe's expressions of the Lagrange planetary equations.[14,15] This method is evaluated elsewhere.[16] The present author recently proposed a method based on approximate stability criteria[17] enabling one to obtain the launch window map with good accuracy without integrating over the lifetime, thereby reducing the formidable computing requirements (formerly as much as 50 hr to 100 hr/yr of lifetime, for one nominal initial orbit and one year of possible launch days) by a factor of about 10^4.

The orbits considered in the preceding studies had rather moderate inclinations on the equator ($\simeq 30°$) and in many instances qualitative insight into the problem could be gained by considering as a crude approximation the planar problem[8] (the orbital planes of sun, moon, and satellite are coincident). At the present time, highly eccentric ($e \simeq 0.95$) orbits with velocity vector, and therefore satellite spin axis, normal to the ecliptic are projected by NASA for several satellites of the IMP-series. These orbits are normal to the plane of the ecliptic and the problem is anything but planar. Using Lidov's theory of the planetary perturbations caused by gravitational fields of other bodies[18] and the complementary concepts of intermediate- and very-long-term stability,[17] this paper attempts to define the significant parameters of, and their influence on, the practical stability of such orbits having prescribed minimum lifetime L.

Approximate Stability Criteria

If the disturbing bodies (moon and sun) are "frozen" at the instantaneous position they assume in inertial space at the time the satellite passes at apogee, it has been found by Lidov[18] that the orbit is of constant semi-major axis, to first ($k = 3$) and second ($k = 4$) order in $(a/p_d)^k$. Therefore studying the evolution of r_P amounts to studying the time-history of eccentricity, which should decrease over the first orbit and never rise above its initial value over $0 \leqslant t \leqslant L$. The method requires that six criteria be checked (some weaker forms of the others, so that there exist only 3 strict criteria) in order to insure, from the configuration at injection and for the point (day of launch, hour of launch) considered, that stability is satisfied in the following ranges:

1) Long-term (characteristic time τ_d)

$$(\delta e)_{LR} = e\epsilon^{1/2}(A_S \sin^2 i_S \sin 2\omega_S + A_M \sin^2 i_M \sin 2\omega_M)/4 \leqslant 0 \quad (1)$$

with

$$A_d = 15\pi(\mu_d/\mu_E)(a/p_d)^3 \epsilon_d{}^{3/2}$$

2) Short-term (characteristic time τ, a few days)

$$(\delta e)_{SR} = -e\epsilon^{1/2}\{(A_M/\epsilon_M{}^{3/2})\beta_{3,M} + (A_S/\epsilon_S{}^{3/2})\beta_{3,S}\} \leqslant 0 \quad (2)$$

in which

$$\beta_{3,d} = \xi_{1,d}\xi_{2,d}(p_d/r_d)^3$$

3–5) Intermediate-term (characteristic time $\tau_d/2$), so that the waviness of the curve of height of perigee vs time about its trendline is limited

$$(\delta e)_{INT} = (\delta e)_{\langle SR,S\rangle,M} + (\delta e)_{LR,M} \leqslant 0 \quad (3)$$

in which $\langle SR,S\rangle,M$ means the short range effect of the sun averaged over τ_M

$$\delta e_j{}^* = -e\epsilon^{1/2} \sum_{k=1}^{j} \left[\frac{A_M}{\epsilon_M{}^{3/2}}\beta_{3,M}{}^{(k)} + \frac{A_S}{\epsilon_S{}^{3/2}}\beta_{3,S}{}^{(k)}\right] \leqslant 0 \quad (4)$$

in which j is the number of passages at perigee over a lunar month. Criterion 5 is: satisfied if $\delta e_{\langle SR,S\rangle,M} > \delta e_{LR}$; if $\delta e_{\langle SR,S\rangle,M} < \delta e_{LR}$, it requires,

$$\delta e_{LR} < [4/9(3)^{\frac{1}{2}}][\delta e_{\langle SR,S\rangle,M} - (\delta e_{LR})] \quad (5)$$

6) Very-long-term, so that e does not assume again its original value after less than the required lifetime L. Thus

$$T^* \geqslant L \quad (6)$$

in which T^*, "predicted lifetime," is computed as indicated in Ref. 19 and recalled in a following section.

Simplified Model: The Two Disturbing Bodies Are in the Same Plane

As an approximation, considering the smallness of the angle between the orbital plane of sun and moon ($I_M = 5.145°$), we take these to be coincident. i_d, inclination of the satellite orbit on the orbital plane of body d, is approximately 90°, or $i_S \simeq i_M \simeq 90°$ (Fig. 1).

Long-Range and Very-Long-Range Stability

The long-range evolution of the orbital parameters in the case of one disturbing body has been described by Lidov.[18] The preceding simplification allows us to apply Lidov's analysis to the present case, provided A is taken to be $A_M + A_S$ (note that A_M/A_S is independent of a and equal to 2.18). Thus the total changes per satellite orbit due to sun and moon

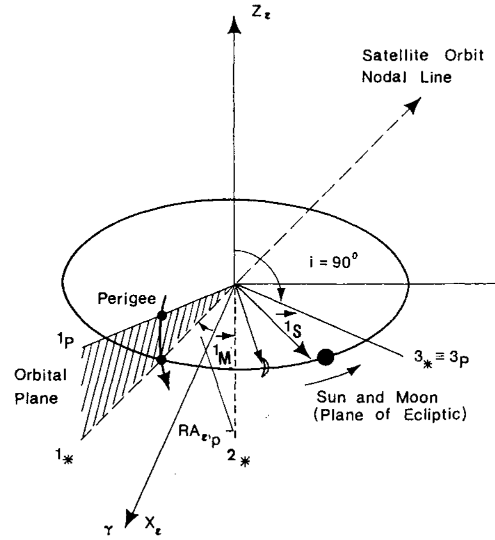

Fig. 1 Geometry of simplified problem.

are (angles, referred to the ecliptic, are not subscripted)

$$\delta a = \delta i = \delta\Omega = 0 \quad (7)$$

$$\delta e = e\epsilon^{1/2}(A_M + A_S)\sin 2\omega/4 \quad (8)$$

$$\delta\omega = [(A_M + A_S)/20]\zeta \quad (9)$$

where

$$\zeta = (5\cos 2\omega - 1)[1 - c_2''/(5\cos 2\omega - 1)]^{1/2} \quad (9a)$$

in which

$$\epsilon = 1 - e^2, \text{ and } c_2'' = (1 - \epsilon_o)(5\cos 2\omega_o - 1).$$

To this approximation, orbits initially normal to the ecliptic will remain so for all time and will have a constant longitude of nodes. Lidov's constants c_1 and c_2, in the system of differential equations to first-order in the perturbations, are (Fig. 2)

$$c_1 = \epsilon \cos^2 i = 0 \quad (10)$$

$$c_2 = (1 - \epsilon)(2/5 - \sin^2\omega) = c_2''/10 \quad (11)$$

Any of the orbits considered will be represented by point $(c_2,0)$ on segment AC of the c_2 axis. From Eqs. (7–9), the evolutions of ω and e are described by Lidov's discussion in which A_d is replaced by $A = A_M + A_S$. Let $\omega_1{}^* = \frac{1}{2}$ arcos $(\frac{1}{5}) = 39.23°$, $\omega_2{}^* = \pi - \omega_1{}^*$, $\omega_3{}^* = \pi + \omega_1{}^*$, $\omega_4{}^* = 2\pi - \omega_1{}^*$.

Then, considering that $\delta e = -A(1 - \epsilon)\epsilon^{1/2}\sin 2\omega/2$, one obtains: 1) If $\omega_4{}^* < \omega < \omega_1{}^*$ (Region 1, Fig. 3), or $\omega_2{}^* < \omega < \omega_3{}^*$ (Region 3, Fig. 3), ω increases. If initially $\omega_o < 0$

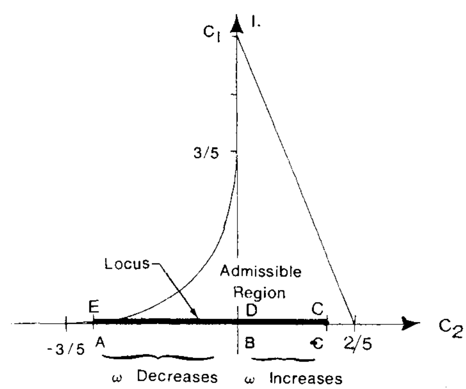

Fig. 2 Lidov's diagram.

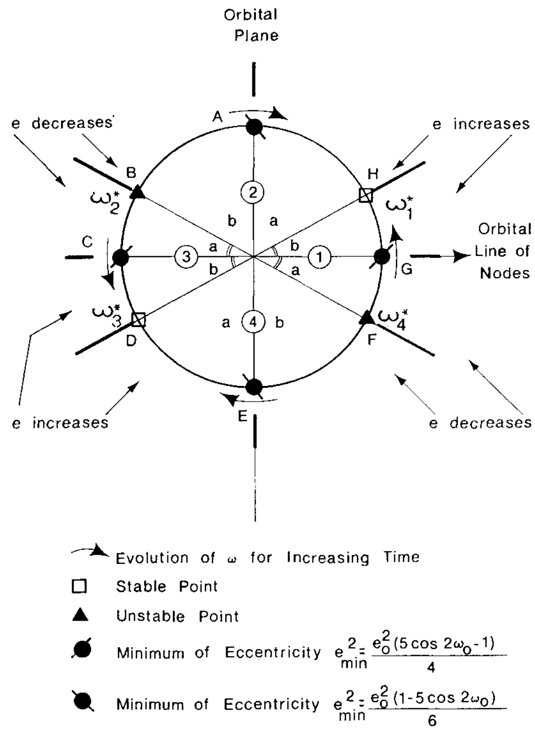

Fig. 3 Evolution of ω and e with time.

($\omega_o < \pi$), e decreases until ω reaches $0(\pi)$. The minimum for e is given by

$$e_{\min}^2 = e_o^2(5 \cos 2\omega_o - 1)/4 \tag{12}$$

Thereafter e increases and reaches $e_{\max}$ at $\omega_{e,\max}$ obtained from

$$1 - e_{\max}^2 = 1 - [1 - (R_E + h_P^*)/a]^2 \tag{13}$$

$$5 \cos 2\omega_{e,\max} - 1 = (1 - \epsilon_o)(5 \cos 2\omega_o - 1)/e_{\max}^2 \tag{14}$$

and $\omega_{e,\max}$ is in the same region as ω_o. If initially $\omega_o > 0$ ($\omega_o > \pi$), there is never a decrease in eccentricity. Thus, from the viewpoint of long-range-stability, we may conclude that sub-regions 1a, 3a are acceptable.

2) If $\omega_1^* < \omega < \omega_2^*$ (Region 2, Fig. 3), or $\omega_3^* < \omega < \omega_4^*$ (Region 4, Fig. 3), ω decreases. If initially $\omega_o > \pi/2(\omega_o > 3\pi/2)$, e decreases until $\omega = \pi/2(\omega = 3\pi/2)$. The minimum for e is

$$e_{\min}^2 = e_o^2(1 - 5 \cos 2\omega_o)/6 \tag{15}$$

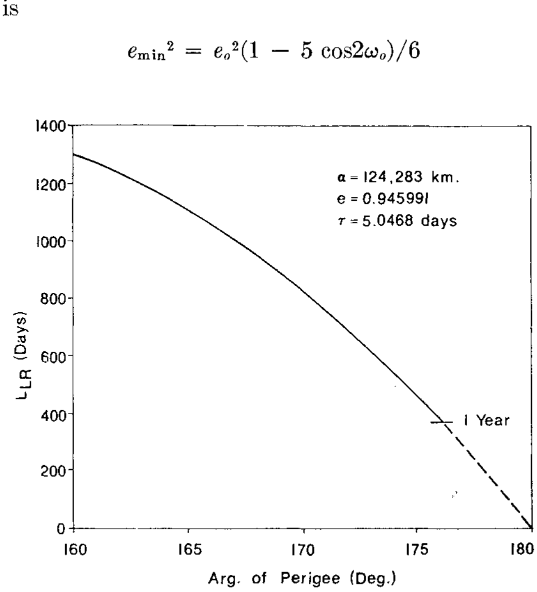

Fig. 4 Long-range lifetime vs argument of perigee at injection.

Thereafter e increases until the value given by Eq. (13). If initially $\omega_o < \pi/2(\omega_o < 3\pi/2)$, the eccentricity is always increasing. Thus, from the viewpoint of long-range stability, regions 2b and 4b are acceptable.

By reason of symmetry, and since our special interest lies in orbits with southwards injection, we restrict the analysis to the second and third quadrants of the orbital plane (Fig. 1). On Fig. 3, B corresponding to $\omega = \omega_2^*$ appears as an unstable point, and D for which $\omega = \omega_3^*$, as a stable point. Only 2b and 3a are acceptable for long-range stability in these two quadrants. Considering very-long-range stability, an assessment of the orbital lifetime can be made here provided it is fairly large compared to the periods of the disturbing bodies (in practice, the required lifetime equals many orbital periods of the moon, but only one or a few orbital periods of the sun). This "long-range" lifetime reads, in satellite periods: Region 2b ($\pi/2 < \omega_o < \omega_2^*$)

$$\left(\frac{L}{2}\right)_{LR} = \left(\frac{20}{A}\right) \int_{\omega_o}^{\pi/2} \zeta^{-1} d\omega \tag{16}$$

Region 3a ($\omega_2^* < \omega_o < \pi$)

$$\left(\frac{L}{2}\right)_{LR} = \left(\frac{20}{A}\right) \int_{\omega_o}^{\pi} \zeta^{-1} d\omega \tag{17}$$

[where ζ is given by Eq. (9a)] and as an example, is given in days vs ω in Fig. 4, for $e_o = 0.945991$, $a = 124{,}283$ km (Region 3a, neighborhood of the ecliptic).

If the orbit originates at $\omega = \omega_2^* \pm \eta$ (η small and positive angle), evolution $B_- \rightarrow A$ will lead to a larger L_{LR} than evolution $B_+ \rightarrow C$: if X is the point having argument $\pi - 2\omega_1^* = 101.54°$, the time spent along B_-X is the same as that along B_+C. Thus

$$t_{B\rightarrow A} = t_{B\rightarrow X} + t_{X\rightarrow A} > t_{B\rightarrow C} \tag{18}$$

In conclusion, larger lifetimes will be possible in region 2b, the larger the closer the argument of perigee is to ω_2^*. In region 3a, for maximum L_{LR}, ω should be made equal to $\omega_2^* + 0$. The upper limit for L_{LR} in this region is infinite since $L_{LR} \rightarrow \infty$ when $\omega \rightarrow \omega_2^*$.

Short-Range and Intermediate-Range Stability

The short-range behavior of the eccentricity, determining the short-range stability, is described by Eq. (2)

$$\delta e_{SR} = -e\epsilon^{1/2}[(A_M/\epsilon_M^{3/2})\beta_{3,M} + (A_S/\epsilon_S^{3/2})\beta_{3,S}] \tag{19}$$

It is apparent that if ω were 180° (Fig. 1), as is the case for an injection at perigee in the ecliptic plane, $\xi_{2,d}$ would vanish and initially short-term stability would be neutral (Criterion 3). Because of the short-range increase of ω, as given by

$$(\delta\omega)_{\omega_o=\pi} = (3/5)[(A_M/\epsilon_M^{3/2})(p_M/r_M)^3 + (A_S/\epsilon_S^{3/2})(p_S/r_S)^3]\epsilon^{1/2} > 0 \tag{20}$$

there is, however, intermediate-term instability (Criterion 4) since $\xi_{2,d}$ would become < 0 at the next orbit. In the quadrants considered, for short-term stability, it is required that the perigee be above the ecliptic plane

$$\pi/2 < \omega < \pi \tag{21}$$

which also guarantees LR stability (Criterion 1)

$$\delta e_{LR} = 1/4e\epsilon^{1/2}A \sin 2\omega < 0 \tag{22}$$

For intermediate-term stability, a margin (of the order of a few degrees) should be provided so that the perigee is sufficiently above the plane of the disturbing bodies. In order to obtain actual lifetimes which are as long as possible, one could possibly select those launch days in the year leading to a slope, on the e vs time from launch curve and over a time of the order of $\tau_S/4$, which is as small as possible. For fixed e_o, $i = \pi/2$, ω_o and thus fixed Lidov's constants c_1 and c_2,

$(\delta e_{LR})_M$ is fixed and one requires to make $\langle \delta e_{SR,S} \rangle, (\tau_S/4)$ as small as possible. As an example, for posigrade orbits, the inclination on the equator of orbits normal to the ecliptic is comprised between $\pi/2 - \epsilon_*$ and $\pi/2$, and to each $\pi/2 - \epsilon_* < i_\alpha < \pi/2$ corresponds one celestial longitude RA_ϵ of the injection vector, in the fourth quadrant of plane (X_ϵ, Y_ϵ) (the second quadrant was rejected for IMP-G due to the earlier occurrence of long periods of shadow). This is represented in Fig. 5. The best launch day(s) of the year will be that for which the unit vector to the sun is along axis $3_P = 3_*$ normal to the orbit, on the average over half a solar modulation, $(\tau_S/4)$. More details are given in what follows.

Conclusions from Simplified Model

In summary, to the approximation of the simplified model and restricting the analysis to the second and third quadrants of the orbital plane (without loss of generality), it is concluded that: 1) the highest realizable value in region 3a for the long-range assessed value of the lifetime is defined by Eq. (17), with ω_o the minimum feasible ω; 2) short-term and long-term stability are strictly realized for ω in the second quadrant, whereas intermediate-term stability requires that ω differ from π (or $\pi/2$) by a negative (positive) margin, in practice a few degrees for a lifetime of one year; and 3) it is possible to define a best day in the year leading to maximum actual lifetime for given e_o, ω and inclination on the equator.

Effect of the Inclination of the Moon's Orbit on the Ecliptic

Let us assume that the satellite orbital plane is still normal to the ecliptic, and let us call $RA_{\epsilon,P}$ the celestial longitude of perigee (Fig. 6). Provided that P is sensibly in the plane of the ecliptic and is above the moon's plane (of longitude of nodes $\Omega_{\epsilon,M}$ referred to the ecliptic)

$$\Omega_{\epsilon,M} - \pi < RA_{\epsilon,P} < \Omega_{\epsilon,M} \tag{23}$$

stability is assured in the long-range, since $(\delta e)_{LR,S} = 0$; $(\delta e)_{LR,M} < 0$. The locus of southernmost (northernmost) admissible perigee points on the unit sphere is obtained by writing

$$(\delta e)_{LR} \approx (1 - \epsilon)\epsilon^{1/2}(A_M\psi_M + A_S\psi_S) = 0 \tag{24}$$

$$\psi_d = \pi - \omega_d \tag{25}$$

or is approximately the arc of great circle having normal $Z_\sim$ inclined by $\tilde{i} = [A_M/(A_M + A_S)]I_M$ on the normal to the ecliptic Z_ϵ, in plane (Z_ϵ, Z_M). In the long range, therefore, I_M may be accounted for by rotating the ecliptic by $\tilde{i}$ about the nodal line of the moon. This new plane of reference $\bar{\omega}_\sim$ then replaces ϵ (the ecliptic). As an example, for a nominal injection at perigee in the plane of the ecliptic and in the fourth quadrant of (X_ϵ, Y_ϵ) (Fig. 6), it is seen that an inclination on the equator $i_\alpha = 90°$ should maximize the long-range-assessed lifetime if $\Omega_{\epsilon,M} = 0°$ (early spring 1969), in the interval $\pi/2 - \epsilon_* < i_\alpha < \pi/2$. (An example follows). It should also be noted that plane $\bar{\omega}_\sim$ will accompany the moon's nodal line in its regression.

In the short range, were the sun alone, condition (21) would still hold, i.e., a perigee in the second quadrant is favorable. For a nominal injection (at perigee, in the ecliptic plane), the short-term effect due to the sun alone is zero and the moon critically determines short-term stability. If the projection of the perigee vector **OP** on the ecliptic is normal to the line of nodes of the Moon N_M referred to the ecliptic in Fig. 6, the Moon is certainly favorable or neutral in the short term if

$$\pi/2 + I_M < \omega_\epsilon < \pi + I_M \tag{26}$$

If the projection of **OP** on the ecliptic coincides with N_M in Fig. 6, or in the general case, the condition for the moon to be

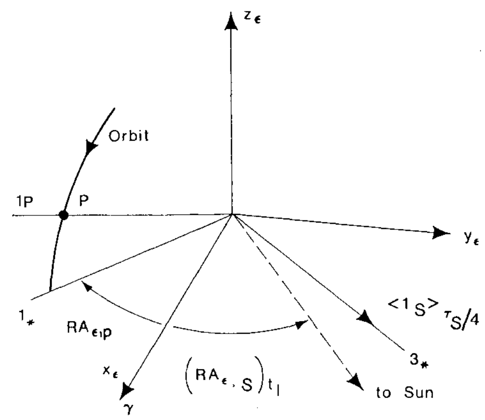

Fig. 5 Geometry of best launch day.

favorable or neutral is

$$\xi_{1,M}\xi_{2,M} > 0 \tag{27}$$

and cannot be satisfied throughout the lunar month.

In the intermediate range, the margin on ω, to which it was referred above should now be construed with reference to plane $\bar{\omega}_\sim$. The condition on the best day of launch still holds approximately, to $O(I_M^2)$.

Effect of a Small Departure of the Orbital Inclination from Normal to the Ecliptic

Having defined a nominal orbit as one normal to the ecliptic, one should now qualify the effects of a slight departure Δi_ϵ of the orbit on the normal to the ecliptic, for instance, as caused by the Earth's rotation for a launch slightly earlier or later than nominal. Lidov's formulas for the long-term changes[18] are written to $O(\Delta i_\epsilon^2)$, $\delta\epsilon_{LR} = -A(1 - \epsilon)\epsilon^{1/2}\sin 2\omega/2$. Now $(\delta\epsilon)_{LR} > 0$ for long-range stability, or $\sin 2\omega < 0$. To the same approximation,

$$(\delta\omega)_{LR} = A\epsilon^{1/2}[2/5 - \sin^2\omega]/2 \tag{28}$$

Therefore, the developments of the two previous sections concerning long-range stability apply.

The orbital inclination in the long-range varies according to

$$(\delta i)_{LR} = -A(1 - \epsilon)\sin 2\omega\Delta i_\epsilon/(4\epsilon^{1/2}) \tag{29}$$

i will increase (decrease) for $\Delta i_\epsilon > 0$ or $i < \pi/2(\Delta i_\epsilon < 0$ or $i > \pi/2)$ and tend to $\pi/2$ for stable orbits.

The rotation of the line of nodes will be of order Δi_ϵ,

$$(\delta\Omega)_{LR} = -A\Delta i_\epsilon[(1 - \epsilon)\sin^2\omega + \epsilon/5]/(2\epsilon^{1/2}) \tag{30}$$

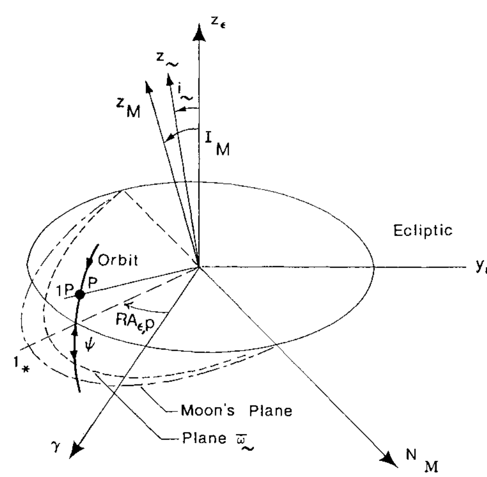

Fig. 6 Geometry of plane $\bar{\omega}_\sim$.

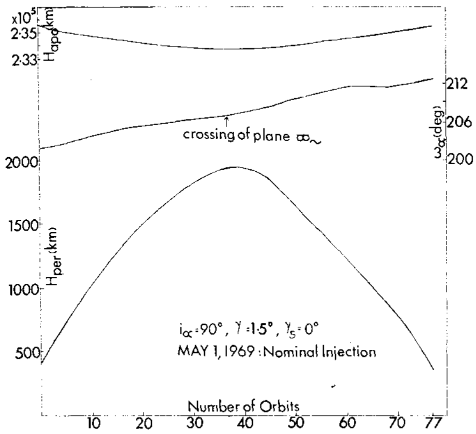

Fig. 7 Orbital elements vs time.

Developments relating to short- and intermediate-term stability involve geometrical conditions in the orbital plane, and still apply to $0(\Delta i_\epsilon^2)$.

A Priori Prediction of Orbital Lifetime

In the determination of launch windows for highly eccentric satellites by the approximate criteria method, it is of great importance to be able to predict quite accurately the orbital lifetime L. Launch hours and days leading to a time in orbit smaller than the required one will be rejected as unstable. In Ref. 17, this was done by assuming that the eccentricity vs time from launch curve could be approximated by a half-sine curve following

$$(e_{\max} - e)/(e_{\max} - e_{\min}) = \sin(\pi t/\tau_{VLR}) \tag{31}$$

with τ_{VLR}, "very-long-range" period, unknown. Here $e_{\max}$ and $e_{\min}$ result from Lidov's constants, c_1 and c_2, computed as if the moon were acting alone (more exactly, as if the sun's apparent orbit was in the plane of the moon's orbit). The superimposed effect of the sun was included in computing the slope δe_{INT} of Eq. (3) to the eccentricity curve at $e = e_o$. As might be expected, this procedure is not sufficiently accurate for small $\pi - \omega$ since it is no more valid to write $\omega_S \approx \omega_M$ [here $(i)_S \approx (i)_M$]. If applied to satellite orbits whose perigee motion, during a significant fraction of the lifetime, occurs between the orbital planes of sun and moon (under the ecliptic and above the moon's plane, say), it leads to predicted lifetimes which are systematically in excess of actual values. This is the result of overlooking, in the calculations, the unfavorable influence of the sun on the evolution of the eccentricity (or height of perigee). This inaccuracy in the contour of the map was of the order of 0.3 hr, too large an error in the case of IMP-G (see last section). In view of the

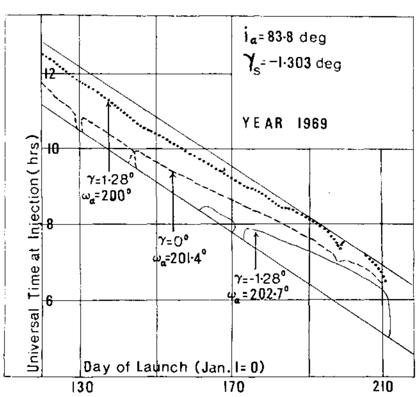

Fig. 8 Effect of flight-path angle at injection.

Table I Comparison of predicted and actual Lifetimes

Day 1969	Inj. hour U.T.	γ, deg	γ_s, deg	Life days pred.	act.	Int. prog.[a]
06/01	9.488	1.5	−1.3	413	410	VP
06/01	9.988	1.5	−1.3	340	370	VP
06/14	9.321	−1.28	−1.3	425	404	VP
					397	EM
05/01	11.363	1.5	0	362	389	VP
05/01	12.263	1.5	0	319	348	VP
05/08	10.097	1.5	0	FAIL[b]	364	VP
05/08	10.297	1.5	0	355	369	VP

[a] VP: Method of variation of parameters; EM: Encke's method
[b] At T + 0.1h:350

small value of the slope $\delta e/\delta\tau$ and accordingly, the small range of variation of the eccentricity when the lifetime is close to one year (typically, $0.946 > e > 0.934$) an alternative approach was taken for Criterion 6. Letting $\omega_\sim$ be the argument of perigee referred to the above defined plane $\bar{\omega}_\sim$, the eccentricity will reach a minimum when the perigee will be contained in $\bar{\omega}_\sim$, i.e., after half the lifetime $L/2$ (Fig. 6). For small $\sin\omega$, it may be written in the case of an injection southwards

$$L/2 = (\pi - \omega_\sim)/(\delta\omega_\sim/\delta\tau) \tag{32}$$

The time-rate of change of $\omega_\sim$ is computed from $(\delta\omega_\sim/\delta\tau) = (\delta\omega/\delta\tau)_{LR,M} + (\delta\omega/\delta\tau)_{LR,S}(1 + w)$ in which, from Lidov's formulas in the long-range

$$(\delta\omega/\delta\tau)_{LR,d} = A_d[(\cos^2 i_d - \epsilon)\sin^2\omega_d + 2\epsilon/5]/(2\epsilon^{1/2}) \tag{33}$$

Factor $1 + w$ results from averaging the expression between brackets in the short-range effect of the sun (with $e_S \simeq 0$)

$$(\delta\omega/\delta\tau)_{SR,S} \simeq (A_S/5)\epsilon^{1/2}[4\xi_{1,S}^2 - \xi_{2,S}^2 - 1] \tag{34}$$

The average is taken over half a solar modulation ($\tau_S/4$) to obtain the linear trend of $\omega_\sim$ vs time. $\xi_{2,S}^2$ can generally be neglected in Eq. (34). It also results, that, for an injection in the neighborhood of the ecliptic, a "best" day for given i_α is one which minimizes $\delta\omega_\sim/\delta\tau$, or for which $(RA_{\epsilon,S} - RA_{\epsilon,P})_{INJ}$ is $5\pi/4 \pm k(\pi/2)$ (k positive integer or zero). An example is given in the last section. In an application to

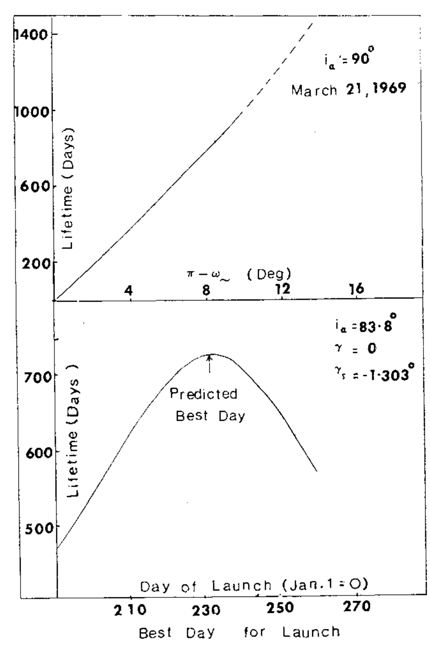

Fig. 9 Variation of lifetime with $\bar{\omega}_\sim$ and day of launch.

IMP-G, this procedure gave good agreement between the predicted and actual figures for the lifetime from digital computer programs.[19] As an example, Fig. 7 shows the evolution of osculating parameters for a sample IMP-G orbit, obtained from a numerical integration program (EOLA) based on the method of variation of parameters. Oblateness is not included in the analysis. The predicted lifetime was 362 days, the actual one 389 days (Case 4 of Table 1). It is interesting to note that the crossing of plane $\bar{\omega}_{\sim}$ (marked by the arrow) quite accurately corresponds to the topping off of the height of perigee. Other comparative lifetime values for IMP-G as obtained from the present method (implemented in program SABAC) and from numerical integration programs ITEM of NASA[20] (Encke's method) and EOLA are shown in Table 1. The average error is of the order of 5% and on the pessimistic side. Case 7 illustrates an inaccuracy in the definition of the limiting contour. However, the lifetime predicted for the next point on the same launch day (for a step of 0.1 hr) is in good agreement with the actual value.

Effect of Earth's Oblateness on Orbital Lifetime

In the course of this investigation, it was found that orbital lifetimes were significantly enhanced by the effect of the equatorial bulge (J_{20} term in Earth's potential), by up to 20% in the cases under study. So far the Earth's potential has been considered spherical in the present analysis. As is well known, due to J_{20}, there will be no secular changes of the satellite semimajor axis, inclination or eccentricity. The line of apsides (in the orbital plane) and line of nodes (in the plane of the equator) will rotate at rates proportional to $(4\text{–}5\sin^2 i_\alpha)$ and $\cos i_\alpha$, respectively. Since all i_α considered here are higher than critical, $(\delta\omega_\alpha/\delta\tau)_{\text{obl}} < 0$, and in the range investigated its magnitude is maximum when $i_\alpha = 90°$. The Earth's equatorial bulge thus exerts a braking effect on the rotation of the line of apsides, thereby appreciably prolonging the lifetime. The same beneficial effect of oblateness on the stability of natural highly eccentric satellites is mentioned in examples given by J. Kovalevsky[21] and Lidov.[22] Investigations are presently under way for the accurate a priori prediction of the lifetime when the effect of oblateness on this quantity is quite significant. This is the case for stable orbits whose perigee rises very little (10^3 to 2×10^3 km) over the whole lifetime, as opposed to the more frequent occurrence described by B. Shute.[6]

Practical Conclusions and Application to IMP-G

The practical implications of the present study, when expressed for a satellite launched southward into an orbit quasi-normal to the ecliptic and with perigee in region 3a (Fig. 3), are as follows:

a) High celestial latitudes of the perigee are required for the stability in all ranges. They will be the more favorable the closer the argument of perigee referred to $\bar{\omega}_{\sim}$ is to $\pi - (\frac{1}{2})$ arcos $(\frac{1}{5})$. In particular a positive flight path angle will be beneficial, within limits allowable on the drop in perigee height as compared to injection height, and mandatory if the injection is to take place in the close neighborhood of the ecliptic.

b) For a nominal launch (at perigee, in the ecliptic) and fixed e and i_α, it is possible to define a best day in the year giving the longest lifetime.

c) The most suitable inclination on the equator, for a nominal launch, is that corresponding to an orbital ascending node at $+\pi/2$ from the moon's node.

d) If the nodal line of the moon is aligned on the vernal line (the case in early spring 1969), the angular height of perigee above $\bar{\omega}_{\sim}$ is sensibly $(\pi - \omega_{\epsilon,P}) + \psi$ (Fig. 6). Therefore, if $i_\alpha < 90°$, advantage can be taken of the Earth's rotation to increase this angle, and consequently the lifetime, by launching earlier than nominal.

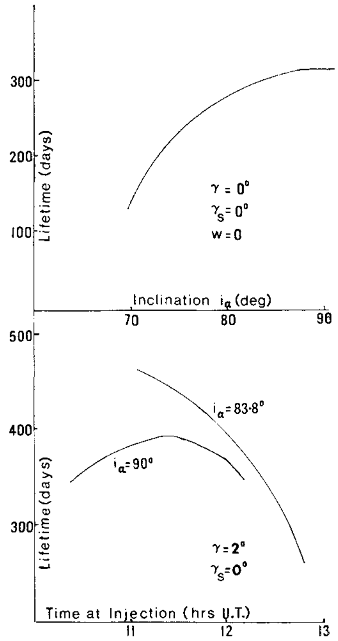

Fig. 10 Variation of lifetime with i_α and hour of injection.

These conclusions can be used with profit in the mission analysis of highly eccentric satellites. Results obtained[19] in the systematic investigation, with use of a UNIVAC 1108, of launch windows for IMP-G ($e = 0.946$; $\tau = 5.04678$ days; $h_A = 235{,}476$ km; $h_P = 334$ km; $(i_\epsilon)_{\text{nom}} = 90°$) indeed confirm the predictions of this study. The investigation included sample checks on stability and lifetime made by reference to two digital programs, one based on Encke's method, the other on a method of variation of parameters. Figure 8–10 are examples. An additional constraint existed which required the alignment of the spin axis (along the inertial velocity vector at injection) on the normal to the ecliptic, within 5°. This already severely limited the launch opportunities to about 1.7 hr on each day, inside which close time limits the stability contour had to be determined. The figures illustrate:

1) the significant improvement in lifetime resulting from a positive flight path angle (or larger $\omega_{\sim}$) in the over-all outlook of the window inside the above referred strip (Fig. 8) and at (nominal) injection, for given i_α (Fig. 9),

2) the good correspondence between the predicted and actual "best day" for fixed e_o, i_α, $\gamma = 0$ (Fig. 9),

3) all other factors being equal, the good agreement between the predicted (89.85°) and actual best inclination at a given date, for a nominal injection and $w = 0$ (Fig. 10), and

4) the improvement of lifetime (Fig. 10) with launch hours that are earlier than nominal for $i_\alpha < 90°$. For $i_\alpha = 90°$, as expected, the lifetime tops off at nominal injection time.

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Alternative Space Transportation Systems for Several Potential National Space Programs

ROBERT SALKELD*

410½ Landfair Avenue, Westwood Village, Los Angeles, Calif.

A general analytical structure for comparing total costs of alternative space transportation system families is formulated against the background of traffic requirements for several potential national space programs. First, payload capabilities and cost characteristics of existing and projected transportation systems are explored to identify which constitute meaningful families. Next, traffic requirements are projected for a range of possible space programs, including cases where emphasis is on near-orbit or far-orbit operations or both, and for different levels of national effort. Then, analytic expressions are written for the total recurring and nonrecurring costs for each system combination, breakeven conditions are determined, and regions of economic preference are plotted, against which are shown projected traffic requirements. The outcome can be quite sensitive to the near-orbit vs far-orbit traffic fraction, as well as to the projected size of the national space program. If only one system is to be developed, there are strong indications that it should be in a medium size range.

Nomenclature

C = total (development plus operational) cost
c = operational cost/lb in near orbit
D = system development cost
F = funded space transportation system
P = projected space transportation system
t = time from initial system operation
W = total weight placed in orbit

Subscripts

f = large (>250,000 lb in near orbit)
l_m = funded system (medium)
f_s = funded system (small)
m = medium (25,000–250,000 lb in near orbit)
p = projected system (large)
pm = projected system (medium)
ps = projected system (small)
s = small (10,000–25,000 lb in near orbit)

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* Consultant. Associate Fellow AIAA.

Introduction

WHATEVER the scope, scale, and growth rates of the national space program beyond the Apollo lunar landing project, it is clear that analytical studies to identify the most economical families of space transportation systems will be a central space planning task. The requirement for such studies was stressed by Heiss and Morgenstern.[1] Recent work in this area, as represented in Ref. 2, has included some treatment of the general question of comparison between alternative systems. It is a complex question, which can